FourierSAT: A Fourier Expansion-Based Algebraic Framework for Solving Hybrid Boolean Constraints

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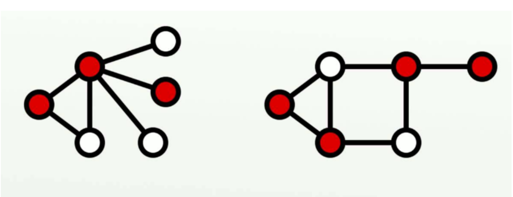
Boolean Constraints and Satisfiability (SAT)

- Boolean variables: $x, y, z, w ... \in \{0,1\}$
- Examples of Boolean Constraints:
  - CNF constraints: $x \vee y \vee z$
  - XOR constraints: $x \oplus y \oplus z$
  - Cardinality constraints: $x + y + z + w \geq 2$
  - Not-All-Equal constraints: $\neg(x \leftrightarrow y \leftrightarrow z \leftrightarrow w)$
- Boolean Satisfiability (SAT) Solving: finding a solution of a system consists of Boolean constraints.

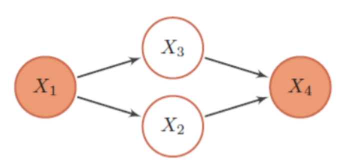
$$\begin{cases} x \vee y \vee z \\ x \oplus y \oplus z \\ x + y + z + w \geq 2 \\ NAE(x, y, z, w) \end{cases}$$

$x = y = z = 1, w = 0$ is a solution!

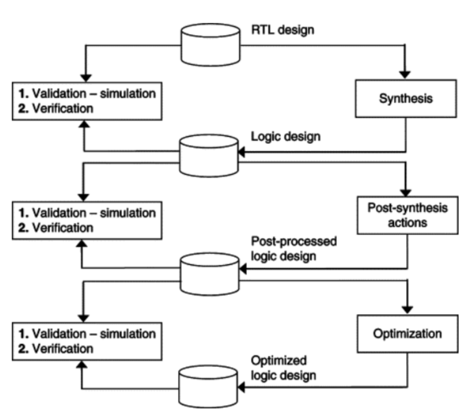
Applications of SAT



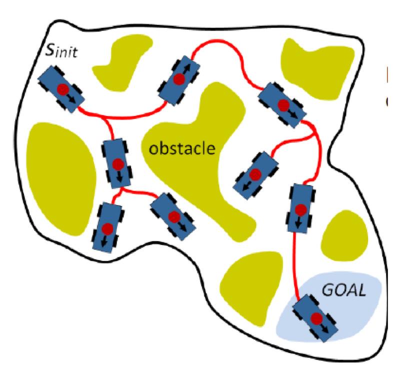
SAT can represent a lot of discrete optimization problem, e.g. minimum vertex covering and vertex coloring in graph theory.



SAT can be used to solve discrete integration problem such as computing the partition function, a fundamental problem in AI.



SAT solver is a critial tool in hardware design and software verification.

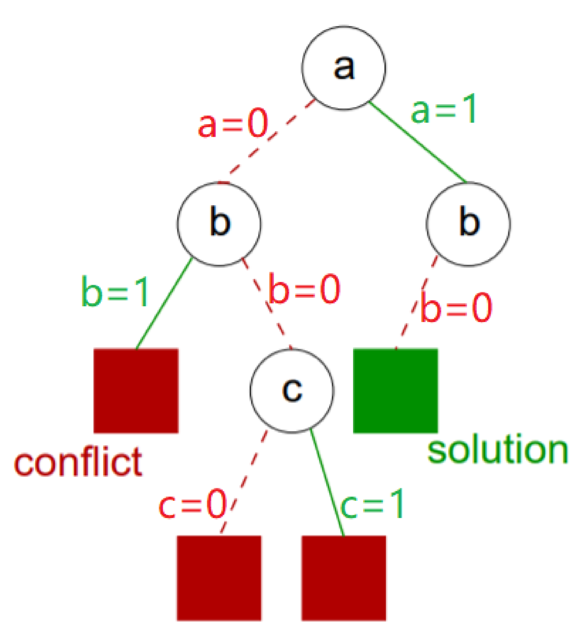


SAT can represent problems in motion planning problems from robotics

Previous Methods

- Searching with backtracking

$$\varphi = (a \vee \neg b \vee d) \wedge (a \vee \neg b \vee e) \wedge (\neg b \vee \neg d \vee \neg e) \wedge (a \vee b \vee c \vee d) \wedge (a \vee b \vee c \vee \neg d) \wedge (a \vee b \vee \neg c \vee e) \wedge (a \vee b \vee \neg c \vee \neg e)$$



- Local search
  Flip one bit "greedily" with most satisfied constraints.

Motivation

There is no general versatile SAT solver that can handle different types of constraints!

References

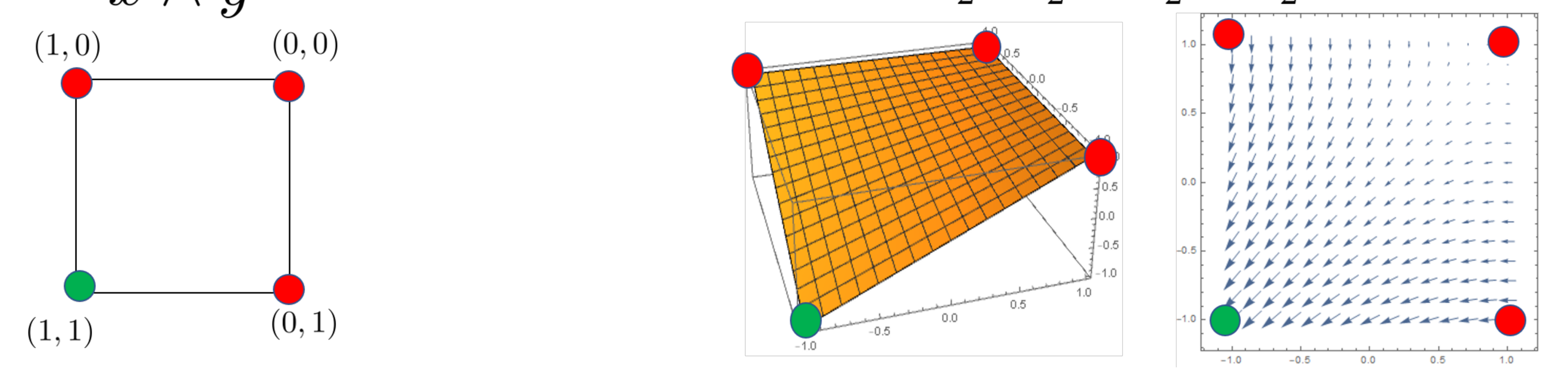
1 Rong Ge, Furong Huang, Chi Jin, Yang Yuan Escaping From Saddle Points — Online Stochastic Gradient for Tensor Decomposition

2 Simons Institution — Bridging Continuous and Discrete Optimization Aug. 16 — Dec. 15, 2017

From Discrete to Continuous

By Fourier transform, Boolean formulas can be expanded to multilinear polynomials. Gradient can be computed and continuous optimization methods can be applied.

$x \wedge y$ $\qquad$ $\frac{1}{2} + \frac{1}{2}x + \frac{1}{2}y - \frac{1}{2}xy$



Let C be the set of constraints and $|C| = m$,

$$Obj(x) = \sum_{c \in C} \texttt{FourierExpansion}(c)$$

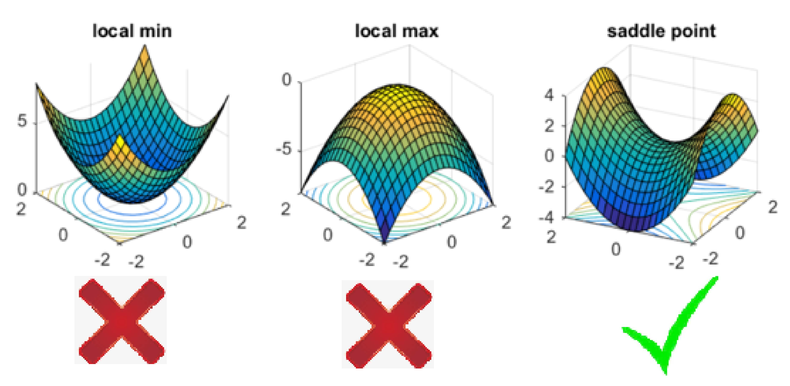
Reduction from SAT to Minimization

Theorem 1

$$c \text{ is satisfiable} \Leftrightarrow \underset{x \in [-1,1]^n}{Min} \texttt{FourierExpansion}(x) = -1$$

$$C \text{ is satisfiable} \Leftrightarrow \underset{x \in [-1,1]^n}{Min} Obj(x) = -|C|$$

The landscape of the curve contains only saddles instead of local min and local max.



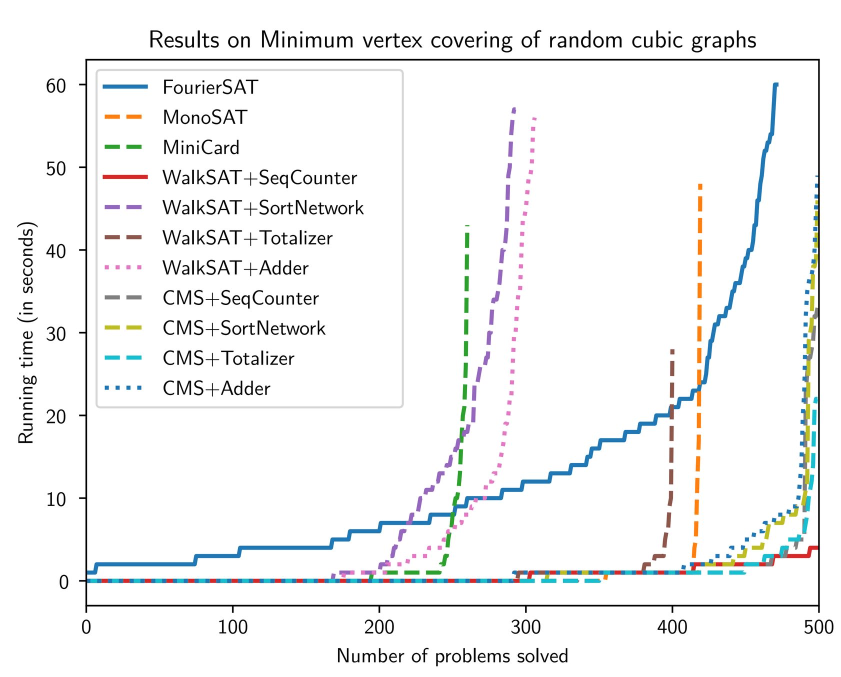
Connecting Continuous and Discrete: Expectation

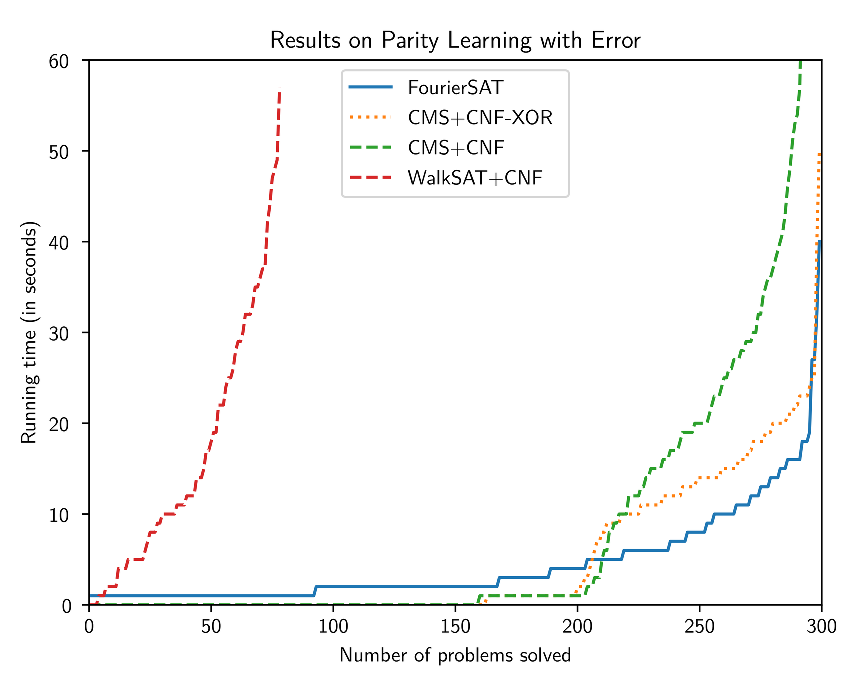
Theorem 2 *Let $a \in [-1,1]^n$ be a continuous assignment, there exists a randomized rounding R such that $R(a) \in \{-1,1\}^n$ and $\mathbb{E}[\#_{SAT}(R(a))] = \frac{m - Obj(a)}{2}$*

Convergence Rate

Theorem 3 *For a formula with n variables and m constraints, Gradient Descent converges to a ϵ-first order critical point in $O(\frac{nm^2}{\epsilon^2})$ iterations.*

Experimental Results





Conclusion:

- FourierSAT is a versatile SAT solver with interests in both theory and practice.
- SAT solving by algebraic methods is a promising direction.
- More work on "Bridging Continuous and Discrete Optimization" is desirable.